The Universe as a Topological Defect in a Higher-Dimensional Einstein–Yang–Mills Theory

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Abstract

An interpretation is suggested that a spontaneous compactification of space-time can be regarded as a topological defect in a higher-dimensional Einstein-Yang-Mills (EYM) theory. We start with D-dimensional EYM theory in our present analysis. A compactification leads to a $D-2$ dimensional effective action of Abelian gauge-Higgs theory. We find a "vortex" solution in the effective theory. Our universe appears to be confined in a center of a "vortex", which has $D-4$ large dimensions. In this paper we show an example with $SU(2)$ symmetry in the original EYM theory, and the resulting solution is found to be equivalent to the "instanton-induced compactification". The cosmological implication is also mentioned.
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Recently there has been much interest in the study of higher-dimensional theories. This is because the candidates for unified theory can be easily and naturally formulated in higher dimensions [1, 2]. If the theories with more than four dimension are to be taken seriously, a mechanism which brings about dynamical compactiacation of the extra dimensions is needed. Usually the size of the extra space is considered to be of order of the Planck length in order that we cannot observe the extra space experimentally in the editing laboratory. Various mechanisms of compactification are proposed in the literature, rasing at variety of "forces" to curl up the extra dimensions [1, 2].

Rubakov and Shaposhnikov obtained a novel mechanism [3]. They considered a self-interacting scalar theory in higher dimensions and discussed the possibility that "we live inside a domain wall". It is implicitly suggested that there are many three-dimensional "worlds" in the higher dimensions in their

scenario. The possibility of "many worlds" is of great hterest particularly in the cosmological context. However, their analysis did not include the connection to gravity. lt is conceivable that no static solution coupled with gravity can be found and such an energetic structure, which is adequately measured in the Planck mass, will soon collapse. In addition, the existence of elementary scalar field with appropriate self-couplings has no compelling reason supported by unification theories.

On the other hand, it is shown in Ref. [4] that gauge boson-Higgs scalar systems are derived from the dimensional reduction of higher-dimensional Yang-Mills theory. Unfortunately, they consider the dimensional reduction as a mere device to obtain various breaking patterns or Higgs mechanism.

Our scenario for compactification is the following: First we consider that the space-time is partially compactified in higher-dimensional Einstein-Yang-Mills (EYM) theory. Then we obtain an efreetive gauge-Higgs theory. Second, we consider a topologlcally non-trivial structure in the effective theory coupled to Einstein gravity. We suppose that stable static solutions can be constructed in this way.

Because we start with EYM theory and consider the configuration with non-trivial topology, it is hopeful to solve the fermion problem in Kaluza-Klein theory [5]. In fact, a simple example has been found. Simplest effective (neutral) scalar theory can be derived from $SU(2)$ gauge theory compactified onto S^3, three-dimensional sphere [6]. There is an exact analytic solution of "kink" or domain wall. This solution turns out to be equivalent to "instanton-induced compactification" presented by Randjbar-Daemi, Salam and Strathdee [7]. Accordingly, the stability has already been guaranteed in this case.

By imagining the two-step compactification, or "double compactification", we anticipate some kind of phase transition in the early history of the universe. As usual with Higgs potential, the effective potential is deferent on temperature and the deformation of the shape of the potential leads to phase transition [6]. During the phase transition, some topological defects are formed and we were born and live inside one of them.

In this paper we adopt $SU(2)$ gauge theory coupled to Einstein gravity in D dimensional space-time as an example. First we consider compactification on S^2, two dimensional sphere, and then obtain effective Abelian Higgs model in $(D-2)$ dimensions. Next we construct a vortex solution with a unit-quantum of flux. The classical solution is coupled to gravity. Here the term "vortex" means that the topological defect has $(D-5)$ dimensional spatial extension in $(D-3)$ dimensional space; the extension is two dimension less than the space it can move.

In the following analysis, we set $D=8$ in practice. The dimensionality of the resulting flat space-time we live in can be taken as an arbitrary number, but in this paper the dinensionality of the flat space-time is set to four.

We begin with the followhg Einstein-Yang-Mills action

$$S = \int d^8x\sqrt{-g}\left(-\frac{1}{2\kappa^2}R + \frac{1}{4e^2}\mathrm{tr}(F_{MN}F^{MN}) + \Lambda\right)\,. \tag{1}$$

Here $\kappa^2 = 8\pi G$; G is Newton constant; e is a gauge coupling constant; Λ is a cosmologlcal constant. The scalar curvature of S^N with unit radius is defined as $R = +N(N-1)$. The suffices M and N run from 0 to 7.

The gauge symmetry group under consideration is $SU(2)$, This may be regarded as a subgroup of a large unified symmetry group.

The field equations are Yang-Mills equations

$$D_M F^{MN} = \nabla_M F^{MN} + i[A_M, F^{MN}] = 0\,, \tag{2}$$

where the field strength is given by

$$F_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N]\,, \tag{3}$$

and Einstein equations

$$R_{MN} = \frac{\kappa^2\Lambda}{3} g_{MN} + \kappa^2 \left(T_{MN} - \frac{1}{6} T g_{MN} \right)\,, \tag{4}$$

where

$$T_{MN} = \frac{1}{e^2} \mathrm{tr} \left(F_{MN} F^P{}_N - \frac{1}{4} F_{PQ} F^{PQ} g_{MN} \right)\,, \tag{5}$$

and

$$T = T^M_M\,. \tag{6}$$

Here R_{MN} is the Ricci tensor derived from the metric g_{MN}.

To solve the equations coupled to gravity, wetake an ansatz for the form of the metric:

$$ds^2 = ds^2(M_4) + g^2(\rho) d\rho^2 + a^2(\rho) d\psi^2 + b^2(\rho) d\Omega^2(S^2)\,, \tag{7}$$

where $d\Omega^2(S^2) = d\theta^2 + \sin^2\theta d\phi^2$ and $0 \le \psi < 2\pi$, $0 \le \theta < \pi$ and $0 \le \phi < 2\pi$. We assume that the four dimensional space-time we live in admits flat Minkowski metric.

The form of gauge fields on S^2 is assumed to be

$$A_\theta = \Phi_1 \frac{1}{2} \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix} + \Phi_2 \frac{1}{2} \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}\,, \tag{8}$$

$$\begin{aligned} A_\phi = {} & -\Phi_1 \frac{1}{2} \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} + \Phi_2 \frac{1}{2} \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix} \\ & + \frac{1}{2} \begin{pmatrix} 1-\cos\theta & 0 \\ 0 & -(1-\cos\theta) \end{pmatrix}\,. \end{aligned} \tag{9}$$

Here Φ_1 and Φ_2 are functions of the coordinates but independent of the S^2 coordinates θ and ϕ. Further, we assume the "$U(1)$ gauge field" in six dimensional space-time, in general, of the form:

$$A_\mu = A_\mu(x^\mu) \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{10}$$

where $A_\mu(x^\mu)$ depends only on the coordinate of six dimensions. μ runs from 0 to 5. Note that here we use a coordinate basis associated with the metric and not an orthorgonal one.

When the ansatz for the form of gauge configurations is substituted to the Yang-Mills equation (2), the equations of motion for Φ_1, Φ_2 and A_μ closely resemble the equations of notion in Abelian Higgs model considered by Nielsen and Olesen [8]. The equations of motion are

$$D^\mu D_\mu \hat{\Phi} + \frac{1}{b^2}(1 - |\hat{\Phi}|^2)\hat{\Phi} = 0\,, \tag{11}$$

$$\nabla_\mu (b^2 F^{\mu\nu}) + i(\hat{\Phi}^* D^\nu \hat{\Phi} - \hat{\Phi} D^\nu \hat{\Phi}^*) = 0\,, \tag{12}$$

where $\hat{\Phi} = \Phi_1 + i\Phi_2$ and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The covariant derivative is defined as $D_\mu = \nabla_\mu + iA_\mu$ where ∇_μ is the covariant derivative associated with six-dimensional metric $g_{\mu\nu}$.

Of course the equations coincide with theirs when $g^2 = 1$, $a^2 = \rho^2$ and b^2=constant. When gravity is coupled, the solution obtained by Nielsen and Olesen is modified except for near the origin, $\rho = 0$. We impose an ansatz for the form of a vortex solution. They are the following [8]:

$$\begin{aligned} \hat{\Phi} &= |\Phi|(\rho) e^{i\psi}\,, && (13)\\ A_\mu &= 0 \quad \text{except for } A_\psi(\rho) \text{ (function on } \rho)\,. && (14) \end{aligned}$$

Furthermore we assume physically plausible properties to solve the equation: In the limit $\rho \to \infty$, $|\Phi| \to 1$ and $b \to 0$, and at $\rho \to 0$, $g^2 = 1$, $a^2 \to \rho^2$ (usual cylindrical coordinates) and the "magnetic flux" $\int A_\psi d\psi \to 0$.

At last, we find the following solution of vortex-type:

$$\begin{aligned} |\Phi|(\rho) &= \frac{\rho/B}{\sqrt{1+(\rho/B)^2}}\,, && (15)\\ A_\psi(\rho) &= \frac{1}{\sqrt{1+(\rho/B)^2}} - 1\,, && (16)\\ g^2(\rho) &= \frac{1}{(1+(\rho/B)^2)^2}\,, && (17)\\ a^2(\rho) &= \frac{\rho^2}{1+(\rho/B)^2}\,, && (18)\\ b^2(\rho) &= \frac{B^2}{1+(\rho/B)^2}\,, && (19) \end{aligned}$$

where $B^2 = \kappa^2/(2e^2)$ and the natness of the large dimensions realizes provided that $\Lambda = 6e^2/\kappa^4$. These algebraic relations remain unchanged when the dimensionality of the flat space-time left untouched varies.

The flux is quantized as expected:

$$\lim_{\rho\to\infty} \frac{1}{2\pi} \int_0^{2\pi} A_\psi d\psi = -1\,. \tag{20}$$

Note that here we obtain an analytic expression of the solution.

For the natural values of the couplings, B is small oforder of the Planck length. Thus the size or the core of the vortex is nearly the Planck scale. The direction of ρ is contructed by gravitational effect, and then it becomes "compact". At first sight the metric of the solution is a little bizarre, but the geometry of the space spanned by ρ, ψ, θ and ϕ turns out to be one of S^4 after the rewriting of the coordinates.

Next, we examine the energy of the configuration of Yang-Mills fields.

One can find

$$F_{\rho\psi} = -\frac{\rho/B^2}{(1+(\rho/B)^2)^{3/2}}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{21}$$

$$F_{\theta\phi} = \frac{\sin\theta}{1+(\rho/B)^2}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{22}$$

$$F_{\rho\theta} = \frac{1/B}{(1+(\rho/B)^2)^{3/2}}\frac{1}{2}\begin{pmatrix} 0 & -ie^{-i(\phi-\psi)} \\ ie^{-i(\phi-\psi)} & 0 \end{pmatrix}, \tag{23}$$

$$F_{\psi\phi} = \frac{\rho\sin\theta/B}{1+(\rho/B)^2}\frac{1}{2}\begin{pmatrix} 0 & -ie^{-i(\phi-\psi)} \\ ie^{-i(\phi-\psi)} & 0 \end{pmatrix}, \tag{24}$$

$$F_{\rho\phi} = -\frac{\sin\theta/B}{(1+(\rho/B)^2)^{3/2}}\frac{1}{2}\begin{pmatrix} 0 & e^{-i(\phi-\psi)} \\ e^{-i(\phi-\psi)} & 0 \end{pmatrix}, \tag{25}$$

$$F_{\psi\theta} = \frac{\rho/B}{1+(\rho/B)^2}\frac{1}{2}\begin{pmatrix} 0 & e^{-i(\phi-\psi)} \\ e^{-i(\phi-\psi)} & 0 \end{pmatrix}. \tag{26}$$

If one utilizes the vierbeins to treat the suffices, one can immediately see the anti-self duality of the solution. Thus the energy density per unit three-dimensional large spatial volume is given by

$$\frac{1}{4e^2}\int d^4y\sqrt{g(y)}\mathrm{tr}\ F^2 = \frac{1}{4e^2}\left|\int d^4y\mathrm{tr}\ F\tilde{F}\right| = \frac{4\pi^2}{e^2}\,, \tag{27}$$

where $y^m = \{\rho, \psi, \theta, \phi\}$ and $g(y) = \mathrm{det}g_{mn}$.

The stability of the solution is expected from this relation. though the full stability analysis should include the perturbation of the metric and the mixing with the modes from graviton. The relation in the form of the gauge configuration exhibits the equivalence of the well-known solution to the one of the "instanton-induced compactification" [7]. We emphasize, however, that when the technique of the type we showed is applied to an EYM theory with other gauge group and compactification, sub as $SU(m+1)$ on CP^m $(m \geq 2)$ [4] we can types of solutions. We will not explore it further here.

In the model of the present type, we cannot take arbitrary values for coupling constants of the scalar Φ. It can be read from the equation of motion that the effective "Higgs self-coupling" is the same as the gauge coupling up to an adequate normalization [9]. This fact suggests the existence of a static multi-vortex solution or a configuration of n-super-imposed vortices. We study of multi-vortex system is the most important task when wc wish to investigate a cosmological scenario (see later).

To conclude this paper, we should mention the cosmological implication. As mentioned before, we wish to consider "double compactification" as a dynamical phase transition, not merely a technical formulation of compactification. The investigation of the high-temperature phase of the model and the study of dynamical evolution of the scale factors are of great importance if one wants to consider inflation or cosmological aspects.

Another remarkable concept is the possibility of "many worlds". If topological defects can be copiously produced after the phase transition, many lower dimensional "worlds" can form networks. The event of "worlds in collision" may occur in higher dimensional space. In analogy with cosmic strings [10], self-crossing of a vortex as a "world" may lead to a "closed" world. The questions now arise: does it then collapse? If we live in the vortex, how about the effect of the collision or crossing on our world? The above questions must be considered not only by analysing classical solutions of multi-defect system but also by including quantum effects of matter and gravity. The selection or the manifold on which the first-step compactification produces the gauge-Higgs system may also be influenced by quantum effects.

As a variation of the scenario, one can consider that we live outside the defects and then we can find the extra space only inside the topological defect in ow universe. This possibility is also worth studying.

If the solution obtained here is included in some series of solutions, we must find the method to obtain the series of the solutions and eitend the solutions to apply to various physical situations. We want to investigate other gauge configuration in other type of the compact manifolds in future works.

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